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Bohr's phenomenon for analytic functions into the exterior of a compact convex body ☆

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ABSTRACT

Bohr's inequality for the class of analytic functions mapping the unit disk into the exterior of a compact convex body is established. In this general case, the radius obtained is $|z| < 3 - 2\sqrt{2}$. When the compact convex body is the closed unit disk, a sharp radius of $1/3$ is obtained.

1. Introduction

Bohr's inequality states that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in the unit disk U and $|f(z)| < 1$ for all $z \in U$, then

$$\sum_{n=0}^{\infty} |a_n z^n| \leqslant 1 \tag{1.1}$$

for all $z \in U$ with $|z| \leqslant 1/3$. This inequality was discovered by Bohr [7] in 1914. Bohr actually obtained the inequality for $|z| \leqslant 1/6$. Wiener, Riesz and Schur, independently established the inequality for $|z| \leqslant 1/3$ and showed that the bound $1/3$ was sharp [10,15,16]. Other proofs were also given in [11–13]. Boas and Khavinson [6], and more recently Aizenberg [3–5] extended the inequality to several complex variables.

Bohr's inequality drew the attention of operator algebraists after Dixon [8] showed a connection between the inequality and the characterization of Banach algebras that satisfy von Neumann's inequality. Specifically, by using Bohr's inequality, Dixon constructed an example of a Banach algebra that satisfies von Neumann's inequality but is not isomorphic to the algebra of bounded operators on a Hilbert space. Paulsen and Singh [11] extended Bohr's inequality to Banach algebras.

A class of analytic (harmonic) functions in the unit disk U is said to satisfy Bohr's phenomenon if an inequality of type (1.1) holds uniformly in $|z| < \rho_0$, for some $0 < \rho_0 \leqslant 1$, and for all functions in the class.

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In this article, we shall consider the space of functions subordinated to a given analytic function. For definition and details of subordination classes, see for example [9, Chapter 6] or [14, p. 35].

Let f and g be two analytic functions in the unit disk U. A function g is subordinate to f if there exists a Schwarz function φ, analytic in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, satisfying $g = f \circ \varphi$. In particular, when f is univalent, g is subordinate to f when $g(U) \subset f(U)$ and $g(0) = f(0)$ ([9, p. 190], [14, p. 35]). Consequently, when g is subordinate to f, then $|g'(0)| \leqslant |f'(0)|$.

In this sequel the class of all functions g subordinate to a fixed function f is denoted by $S(f)$ and $f(U) = \Omega$. The class $S(f)$ is said to satisfy Bohr's phenomenon if for any $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, there is a ρ_0, $0 < \rho_0 \leqslant 1$, so that

$$\sum_{n=1}^{\infty} \left|b_n z^n\right| \leqslant d\left(f(0), \partial\Omega\right) \tag{1.2}$$

for $|z| < \rho_0$. Here $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of a domain Ω. Obviously, when $\Omega = U$, $d(f(0), \partial\Omega) = 1 - |f(0)|$ and in this case (1.2) reduces to (1.1).

It is known that $S(f)$ has Bohr's phenomenon when f is univalent. Abu-Muhanna [2] recently showed that every $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$ satisfies (1.2) for $|z| \leqslant \rho_0 = 3 - 2\sqrt{2} \cong 0.17157$. The radius ρ_0 is sharp for the Koebe function $f(z) = z/(1-z)^2$.

In particular, when f is convex, it was shown in [5] that (1.2) remains valid for $\rho_0 = 1/3$, a result which includes (1.1) when $\Omega = U$.

In this article, we shall consider the case when Ω is the exterior of a compact convex body, and $\mathcal{F}_\Omega$ is the class of all analytic functions mapping U into Ω. The measure that will be used in this instance is the spherical chordal measure given by

$$\lambda(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

When Ω is the exterior of the closed unit disk U, it is shown in Theorem 2.1 that (1.2) remains valid with $d(f(0), \partial\Omega)$ replaced by $\lambda(f(0), \partial\Omega)$ and $\rho_0 = 1/3$. This radius ρ_0 obtained is sharp. In the general situation when Ω is the exterior of a compact convex body, it is shown in Theorem 2.2 that (1.2) holds with $d(f(0), \partial\Omega)$ replaced by $\lambda(f(0), \partial\Omega)$ and $\rho_0 = 3 - 2\sqrt{2}$. However, the ρ_0 obtained may not be sharp.

We shall require the following results.

Proposition 1.1. *(See [1].) If F is an analytic univalent function mapping the unit disk U onto Ω, where the complement of Ω is convex, and $F(z) \neq 0$, then any analytic function $f \in S(F^n)$, $n = 1, 2, \ldots$, can be expressed as*

$$f(z) = \int_{|x|=1} F^n(xz)\, d\mu(x),$$

for some probability measure μ on the unit circle $|x| = 1$. Consequently,

$$f(z) = \int_{|x|=1} \exp\left(F(xz)\right) d\mu(x), \tag{1.3}$$

for every $f \in S(\exp(F))$.

We shall also require the Koebe one-quarter distortion inequalities

$$1 \geqslant d(0, \partial\Omega) \geqslant \frac{1}{4} \tag{1.4}$$

when f is univalent and normalized by $f(0) = 0$ and $f'(0) = 1$, see for example [9, pp. 32, 45] or [14, pp. 21–22].

2. Subordination to the complement of a compact convex body

First we consider the case when $\Omega = c\overline{U}$, where $c\overline{U}$ denotes the complement of $\overline{U}$. Then any universal covering map is given by

$$\exp\left(\frac{1+\varphi(z)}{1-\varphi(z)}\right),$$

where $\varphi(z) = (z+a)/(1-\bar{a}z)$ is a Möbius transformation.

In this case $\mathcal{F}_\Omega$ consists of all analytic functions mapping the unit disk U into $|w| > 1$. Here is the main result, which generalizes Bohr's theorem from the interior of the disk U to its exterior.

Theorem 2.1. *If $\Omega = c\overline{U} = \{w\colon |w| > 1\}$ and $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in \digamma_\Omega$, then*

$$\lambda\left(\sum_{n=0}^{\infty} |a_n||z|^n, |a_0|\right) \leqslant \lambda(a_0, \partial\Omega)$$

for $|z| \leqslant 1/3$. Moreover, the bound $1/3$ is sharp.

As a preliminary to the proof, the class $\digamma_{c\overline{U}}$ is shown to be the union of subordination classes.

Proposition 2.1. *Any $f \in \digamma_{c\overline{U}}$ is subordinate to some universal covering map $G: U \to c\overline{U}$, with $f(0) = G(0) = a_0$. In other words, $f = G \circ \varphi$, where φ is analytic in U, $|\varphi(z)| < 1$ and $\varphi(0) = 0$.*

Proof. Since $\operatorname{Re}\log f(z) > 0$ in U, it is clear that $\log f$ maps U into the right-half plane. Let

$$b = \frac{\log a_0 - 1}{\log a_0 + 1}$$

and

$$\psi(z) = \frac{z + b}{1 + \bar{b}z}.$$

Then the function

$$W(z) = \frac{1 + \psi(z)}{1 - \psi(z)} \tag{2.1}$$

maps $|z| < 1$ univalently into the right-half plane with $W(0) = \log a_0$. Thus

$$\log f = W \circ \varphi$$

for some analytic φ in U, $|\varphi(z)| < 1$ and $\varphi(0) = 0$. The result now follows by letting

$$G(z) = \exp\bigl(W(z)\bigr). \qquad \square \tag{2.2}$$

Here now is the proof of Theorem 2.1.

Proof of Theorem 2.1. Let G be as given in (2.1) and (2.2), and f be subordinate to G. Write

$$G(z) = a_0\left(1 + \sum_{n=1}^{\infty} B_n z^n\right),$$

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n.$$

Then

$$W(z) = \frac{1 + \psi(z)}{1 - \psi(z)} = (\operatorname{Re}\log a_0)\left(\frac{1+z}{1-z}\right) + i \operatorname{Im}\log a_0 = \log a_0 + \frac{\log|a_0|^2 z}{1 - z},$$

and

$$G(z) = a_0\left(1 + \sum_{n=1}^{\infty} B_n z^n\right) = a_0 \exp\left(\frac{\log|a_0|^2 z}{1 - z}\right). \tag{2.3}$$

It follows from (2.2) and (1.3) that

$$|a_n| \leqslant |a_0 B_n|, \quad \text{for all } n \geqslant 1,$$

and

$$\sum_{n=1}^{\infty} |a_n||z|^n \leqslant |a_0| \sum_{n=1}^{\infty} |B_n||z|^n. \tag{2.4}$$

Now

$$|a_0 B_1| = \left|G'(0)\right| = |a_0|\left|W'(0)\right| = 2|a_0|\frac{1-|b|^2}{|1-b|^2} = 2|a_0|(\operatorname{Re}\log a_0) = |a_0|\log|a_0|^2.$$

Next, we show that the sequence B_n is positive and increasing. It follows from (2.3) that

$$B_1 = \log|a_0|^2 > 0,$$
$$B_2 = \frac{\log|a_0|^2}{2}(B_1+2) = \log|a_0|^2\left(\frac{1}{2}B_1+1\right) = \frac{1}{2}B_1^2 + B_1 > B_1. \tag{2.5}$$

Differentiating G in (2.3) yields

$$G'(z) = \frac{\log|a_0|^2}{(1-z)^2}G(z).$$

Hence

$$\left(1-2z+z^2\right)G'(z) = \log|a_0|^2 G(z).$$

This gives the recurrence relation

$$B_{n+1} = \frac{\log|a_0|^2+2n}{n+1}B_n - \frac{n-1}{n+1}B_{n-1} = \left(2+\frac{\log|a_0|^2-2}{n+1}\right)B_n - \frac{n-1}{n+1}B_{n-1}. \tag{2.6}$$

Clearly (2.5) shows that $B_2 > B_1$. Assuming that $B_n > B_{n-1}$, it follows now from (2.6) that

$$B_{n+1} - B_n = \left(1+\frac{\log|a_0|^2-2}{n+1}\right)B_n - \frac{n-1}{n+1}B_{n-1} = \frac{\log|a_0|^2}{n+1}B_n + \frac{n-1}{n+1}(B_n - B_{n-1}) > 0.$$

Hence the sequence B_n is increasing. Consequently, (2.4) implies that, for $|z| \leqslant \rho$,

$$\sum_{n=0}^{\infty}|a_n||z|^n \leqslant |a_0|\sum_{n=0}^{\infty}B_n\rho^n = |a_0|\exp\left[\frac{\log|a_0|^2\rho}{1-\rho}\right] = |a_0||a_0|^{\frac{2\rho}{1-\rho}}. \tag{2.7}$$

When $\rho = 1/3$, then

$$\sum_{n=0}^{\infty}|a_n||z|^n \leqslant |a_0|^2. \tag{2.8}$$

Simple calculation shows that

$$\frac{\lambda(|a_0|, |a_0^2|)}{\lambda(|a_0|, 1)} = \frac{\sqrt{2}|a_0|}{\sqrt{1+|a_0^4|}} < 1, \tag{2.9}$$

and consequently, it follows from (2.8) and (2.9) that

$$\lambda\left(\sum_{n=0}^{\infty}|a_n||z|^n, |a_0|\right) \leqslant \lambda\left(|a_0|, \left|a_0^2\right|\right) \leqslant \lambda\left(|a_0|, 1\right) = \lambda(a_0, \partial\Omega).$$

For sharpness, assume that $\rho > 1/3$. Then by (2.3) and (2.7),

$$\left|G(\rho)\right| = |a_0|\sum_{n=0}^{\infty}B_n\rho^n = |a_0||a_0|^{\frac{2\rho}{1-\rho}} = |a_0|^{\frac{1+\rho}{1-\rho}}.$$

Note that $\frac{1+\rho}{1-\rho} = 2+\delta$ with $\delta > 0$, and $\frac{1+\rho}{1-\rho} \to 2$ as $\rho \to \frac{1}{3}$. Also note that $\frac{|a_0|^{\frac{2\rho}{1-\rho}}-1}{|a_0|-1} \to \frac{2\rho}{1-\rho} = 1+\delta$ as $|a_0| \to 1$. Hence

$$\frac{\lambda(|a_0|, |a_0|^{\frac{1+\rho}{1-\rho}})}{\lambda(|a_0|, 1)} = \frac{\sqrt{2}|a_0|\frac{|a_0|^{\frac{2\rho}{1-\rho}}-1}{|a_0|-1}}{\sqrt{1+|a_0|^{4+2\delta}}} \to (1+\delta)$$

as $|a_0| \to 1$. Consequently, for $|a_0|$ close to 1,

$$\lambda\left(|a_0|\sum_{n=0}^{\infty}B_n|z|^n, |a_0|\right) = \lambda\left(|a_0|, |a_0|^{\frac{1+\rho}{1-\rho}}\right) > \lambda\left(|a_0|, 1\right) = \lambda(a_0, \partial\Omega). \qquad \square$$

The theorem below gives a result under a more general setting than Theorem 2.1.

Theorem 2.2. *Let Δ be a compact convex body with $0 \in \Delta$, $1 \in \partial\Delta$, and $\Omega = c\Delta$. Suppose the universal covering map from U into Ω has a univalent logarithmic branch that maps U into the complement of a convex set. If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in F_\Omega$ satisfies $a_0 > 1$, then for $|z| < 3 - 2\sqrt{2} \cong 0.17157$,*

$$\lambda\left(\sum_{n=0}^{\infty} |a_n||z|^n, |a_0|\right) \leqslant \lambda(a_0, \partial\Omega).$$

Proof. Let F be the universal covering map from U onto Ω with $F(0) = a_0$. Let $G(z) = \log F(z)$ be its univalent logarithmic branch. Then

$$F(z) = \exp G(z),$$
$$a_0 + \sum_{n=1}^{\infty} A_n z^n = \exp\left(\log a_0 + \sum_{n=1}^{\infty} c_n z^n\right).$$

As G is univalent,

$$\frac{G(z) - \log a_0}{c_1} = g \in S,$$

where S is the class consisting of normalized analytic univalent functions in U. For $|z| \leqslant \rho$, it follows from comparing coefficients that

$$|a_0| + \sum_{n=1}^{\infty} |A_n|\rho^n \leqslant |a_0| \exp\left(\sum_{n=1}^{\infty} |c_n|\rho^n\right).$$

Further, since $g \in S$, then $|c_n| \leqslant n|c_1|$ for each n, and

$$\sum_{n=1}^{\infty} |c_n|\rho^n \leqslant |c_1| \frac{\rho}{(1-\rho)^2}.$$

Hence for $|z| \leqslant \rho$, it follows that

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0| \exp\left(\sum_{n=1}^{\infty} |c_n||z|^n\right) \leqslant |a_0| \exp\left(|c_1| \frac{\rho}{(1-\rho)^2}\right). \tag{2.10}$$

Since $0 \notin G(U)$, then $-\log a_0/c_1 \notin g(U)$. Thus the Koebe one-quarter distortion result (1.4) implies that

$$|c_1| \leqslant 4|\log a_0|,$$

and (2.10) yields

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0| \exp\left(4|\log a_0| \frac{\rho}{(1-\rho)^2}\right).$$

If $a_0 > 1$, then

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0|^{1+\frac{4\rho}{(1-\rho)^2}}. \tag{2.11}$$

Simple calculations show that when $\rho \leqslant 3 - 2\sqrt{2}$, then $4\rho/(1-\rho)^2 \leqslant 1$. Hence (2.11) becomes

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0|^2,$$

and (1.3) yields

$$\lambda\left(\sum_{n=0}^{\infty} |a_n||z|^n, |a_0|\right) \leqslant \lambda\left(|a_0|, \left|a_0^2\right|\right) \leqslant \lambda\left(|a_0|, 1\right) = \lambda(a_0, \partial\Omega). \qquad \square$$

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